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ON CLEAVING A PLANAR GRAPH

Shankar M. VENKATESAN*

Department of Computer Science, University of Minnesota, 136 Lind Hall, 207 Church Street SE, Minneapolis, M.N. 55455, U.S.A.

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We first show that the removal of $4\sqrt{n/\epsilon}$ vertices from an n-vertex planar graph with non-negative vertex weights summing to no more than 1 is sufficient to cleave or recursively separate it into components of weight no more than a given ϵ, thus improving on the $2\sqrt{6}\sqrt{n/\epsilon}$ bound shown in [6]. We then derive worst-case bounds on the number of vertices necessary to separate a planar graph of a given radius into components of weight no more than ϵ.

The computational complexity of many graph algorithms that use some graph separator theorem [4] or other as the basis for their so called 'divide-and-conquer' implementation is much dependent on the size of the separator and on the cost of finding such a separator. For this reason, the study of graph separator theorems appears to be a worthwhile endeavor. In this paper, we look at the problem of separating a weighted planar graph into components none of which exceed a given ϵ in weight. Our first result is that the removal of $4\sqrt{n/\epsilon}$ vertices from any n-vertex planar graph with non-negative vertex weights summing to no more than 1 is sufficient to 'cleave' or recursively separate it into components of weight no more than ϵ. We then study the worst-case bounds on the number of vertices necessary to similarly separate a planar graph as a function of ϵ and of the radius of the graph. The reader is referred to [4, 5] for related results and terminology.

Consider an n-vertex graph G, and let a breadth-first spanning tree T [4] be given for G. Let s be a suitably chosen positive integer, and X_j $(0 \leqslant j < s)$ be the set of all vertices at those levels i in T such that $i \mod s = j$. If the number of levels in T is at least s, then there exists a k such that $|X_k| \leqslant \lfloor n/s \rfloor$ (see also [1, 6]). Let such a set X_k be denoted by L_s (if the number of levels in T is less than s, we let $L_s = \emptyset$). We start with two easy lemmas from [6] on the effect of contracting all the vertices in L_s to a single vertex.

Lemma 1 [6]. *Let G be a planar graph. Then the components of $G - L_s$ can be found as disjoint subgraphs in another planar graph G' which contains $|G| - |L_s| + 1$ vertices and which has radius no more than $s - 1$.*

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Proof. Consider the lowest level p in some maximal component C of $G - L_s$. Contracting the connected subgraph of G induced by the vertices at levels less than p into a new root vertex and discarding the vertices in $G - C$ results in a planar graph of $|C| + 1$ vertices which contains C as a subgraph and has radius no more than $s - 1$. When this operation is performed on all such components C and the new root vertices are identified, we obtain the said graph G'. $\square$

Lemma 2 [6]. *Let G be a planar graph of radius no more than $s - 1$, and let G^* be a connected p-vertex subgraph of G. Then there exists a p-vertex planar graph G' of radius no more than $s - 1$ which contains G^* as a subgraph.*

Proof. Consider a vertex $v \in G - G^*$. Identification of v with an adjacent vertex w results in a planar graph having radius no more than $s - 1$ which contains the entire subgraph induced by the vertices of G^*. When this is done on all the vertices in $G - G^*$, we are left with the said graph G'. $\square$

The proofs of the two previous lemmas are simple, and we have included them here only to make the discussion self-contained. We do not however include here the proofs of the next two lemmas which are more involved, and refer the reader to [4, 2].

Lemma 3 [4]. *Let G be a planar graph of radius s, with non-negative weights on its vertices adding to no more than 1. Then the vertices of G can be separated into sets A, B, C such that neither A nor B has weight exceeding $\frac{2}{3}$, there is no edge in G that connects a vertex in A with a vertex in B, and C contains no more than $(2s + 1)$ vertices, one the root of a shortest spanning tree.*

A *weighted regular γ-partition* (A, B, C, D) [2] of a weighted graph G is a partition of its vertices into four sets A, B, C, D, such that A and B are not connected to each other, B and C are not connected to each other, C and A are not connected to each other, and $weight(A) \leq (1 - \gamma)$, $weight(B) \leq (1 - \gamma)$, and $weight(C) \leq \gamma$. The following lemma is due to Djidjev [2].

Lemma 4 [2]. *Let G be a planar graph of radius s, with non-negative weights on its vertices summing to no more than 1. If $\frac{1}{2} \leq \gamma \leq 1$, then there exists a weighted regular γ-partition (A, B, C, D) of G such that D contains no more than $3s + 1$ vertices, one the root of a shortest spanning tree.*

Theorem 1 [7]. *Let G be an n-vertex planar graph with non-negative vertex weights adding to no more than 1, and let $0 < \epsilon \leq 1$. Then there exists some set C of $4\sqrt{n/\epsilon}$ vertices whose removal leaves G with no component of weight larger than ϵ.*

Proof. We first describe an algorithm with an interesting analysis that finds a set C consisting of no more than $2\sqrt{5}\sqrt{n/\epsilon}$ vertices. This algorithm consists of three stages. In the first stage, we first pick $s = \sqrt{n\epsilon/5}$, find L_s with no more than n/s vertices, modify $G - L_s$ into a graph G' of radius no more than $s-1$ as in Lemma 1, and let $C = L_s$. Next, while there is a component G^* of $G' - C$ with weight more than 2ϵ, we modify the component into one of radius no more than $s-1$ as in Lemma 2, partition it into sets A^*, B^*, C^* as in Lemma 3, and let $C = C \cup C^*$.

In the second stage, we look at each component of $G - C$ in the weight range $(\epsilon, 3\epsilon/2]$, separate it into two components each of weight no more than ϵ by one application of Lemma 3, and add the separator to C. In the third stage, we look at each component of $G - C$ in the weight range $(3\epsilon/2, 2\epsilon]$, and separate it into *three* components each of weight no more than ϵ by one application of Lemma 4 (with $\gamma = \frac{1}{2}$), and add the separator to C.

The analysis of the algorithm above proceeds as follows. Consider the recursive separation (in the first stage) of G' as a directed binary tree with G' as the root, where every *interior* node represents a *parent* component broken up into its two *children* at a cost of $(2(s-1))$ vertices. At the end of the first stage, the parent of every *leaf* node in the tree must represent a component of weight larger than 2ϵ, since the parent is separated by Lemma 3. Charge the *weight* of the separator used at the parent of every leaf node to its two children (one of whom may not be a leaf) in such a manner that the total charge on the lighter of its two children is at least one-third the weight of the parent. Note that the weight of the parent is precisely the weight of its two children plus the weight x of the separator used at the parent; however since x could be as large as the weight of the parent component itself, either or both of the two children may have less than one-third the weight of the parent; this condition has to be avoided if we are to bound the *number* of leaves; hence the charging argument where we assign the weight x of the parental separator vertices to the leaf children so that the total weight of each leaf child is at least one-third the weight of the parent. With this scheme, every leaf node has charge larger than $2\epsilon/3$, and thus there are less than $3/(2\epsilon)$ leaf nodes. Therefore there are less than $3/(2\epsilon) - 1$ interior nodes at the end of the first stage. Since each interior node contributes no more than $2s$ vertices to the separator C by Lemma 3, $|C| \leq n/s + 3s/\epsilon$ at the end of the first stage.

Let the total weight of the components separated in the second stage be y. Then there are no more than (y/ϵ) such components, contributing a total of no more than $2s \cdot y/\epsilon$ vertices to C. Then the total weight of the components separated in the third stage is no more than $1 - y$. Thus there are no more than $2(1-y)/(3\epsilon)$ components separated in the third stage, contributing a total of no more than $2s \cdot (1-y)/\epsilon$ vertices to C. Then

$$|C| \leq n/s + 3s/\epsilon + 2s \cdot y/\epsilon + 2s \cdot (1-y)/\epsilon \leq n/s + 5s/\epsilon \leq 2\sqrt{5}\sqrt{n/\epsilon}.$$

We now proceed to describe the steps of our second algorithm which builds on the ideas from the first and finds a set C of size no more than $4\sqrt{n/\epsilon}$.

1. Pick $s = \sqrt{n\epsilon/4}$, find L_s with no more than n/s vertices, modify $G - L_s$ into a graph G' of radius no more than $s-1$ as in Lemma 1, and let $C = L_s$.
2. Let k be the smallest integer such that $2^{k+1} \geq 1/\epsilon$, and let $i = k$.
3. Let the total weight of all the components of $G - C$ in the weight range $(2^i \cdot \epsilon,\ 2^i \cdot 3\epsilon/2]$ be y_i. Separate each such component into two components each of weight no more than $2^i \cdot \epsilon$ by one application of Lemma 3, and add the separator to C.
4. Then the total weight of all the components of $G - C$ in the weight range $(2^i \cdot 3\epsilon/2,\ 2^{i+1} \cdot \epsilon]$ is no more than $(1 - y_i)$. Separate each such component into *three* components each of weight no more than $2^i \cdot \epsilon$ by one application of Lemma 4, and add the separator to C.
5. Set $i = i - 1$. If $i \geq 0$, go to Step 3, else return.

The algorithm has a straightforward analysis. The number of components separated in Step 3 is no more than $y_i/(\epsilon 2^i)$, and the increase in $|C|$ is no more than $2s \cdot y_i/(\epsilon 2^i)$. The number of components separated in Step 4 is no more than $2(1-y_i)/(3\epsilon 2^i)$, and the increase in $|C|$ is no more than $3s \cdot 2(1-y_i)/(3\epsilon 2^i)$, which is no more than $2s(1-y_i)/(\epsilon 2^i)$. Thus the increase in $|C|$ in Steps 3 and 4 is no more than $2s/(\epsilon 2^i)$. Therefore

$$|C| \leq n/s + \sum_{i=0}^{k} 2s/(\epsilon 2^i) \leq n/s + 4s/\epsilon \leq 4\sqrt{n/\epsilon}. \quad \square$$

The result obtained in Theorem 1 improves on the bound $2\sqrt{6}\sqrt{n/\epsilon}$ obtained in [6], which itself was an improvement over the first such result shown in [5]. However, our proof is quite different from that given in [5] in the sense that we bring down the radius of the original graph in one stroke to a manageably small value, after which we no longer have to change the radius of any intermediate component. The computational significance of our proof method has been recently established by Djidjev [3] who has shown how to implement our approach in *linear* time, thus putting possibly many important combinatorial problems posed on planar graphs in *linear* time.

The result in Theorem 1 is tight upto a constant factor in the worst case, and we state this as a proposition the intuition behind which should be self-evident. We include a sketch of its proof since we cannot find a reference to such a result in the literature.

Proposition. *For every fixed $\epsilon \leq \frac{1}{2}$ and for every $n > 1/\epsilon$, there exists an n-vertex planar graph G such that every set of vertices whose deletion from G leaves only components of size no more than $\epsilon \cdot n$ must have size at least $a_1\sqrt{n/\epsilon}$ for some fixed constant a_1.*

Proof. Consider a $\sqrt{n} \times \sqrt{n}$ grid graph G, and let X be a subset of its vertices whose deletion leaves behind components $C_1, C_2, \ldots, C_k$ of sizes $c_1, c_2, \ldots, c_k$ respectively, where each $c_j \leq \epsilon \cdot n$. Since each C_j is a subgraph of a grid graph, and since $|G - C_j| \geq (1-\epsilon)n \geq n/2$, the number of vertices of X connected to C_j must be at least $a'\sqrt{c_j}$ for some fixed positive constant a'. However, a vertex of X can be connected to at most four such components. Hence the size of X must be at least $\sum_{1 \leq j \leq k} d\sqrt{c_j}$ where $d = a'/4$. Since each c_j is no larger than $\epsilon \cdot n$, the size of X is no smaller than

$$\sum_{1 \leq j \leq (n-|X|)/(n \cdot \epsilon)} d\sqrt{\epsilon \cdot n},$$

whence $|X|(1 + d\sqrt{1/(n\epsilon)}) \geq d\sqrt{n/\epsilon}$. Then $|X| \geq (d\sqrt{n/\epsilon})/(1+d)$. Putting $a_1 = d/(1+d)$, the result follows. □

Theorem 2. *Any planar graph of radius s with non-negative vertex weights summing to no more than 1 can be split into components of weight no more than ϵ by removing a set of no more than $4s/\epsilon$ vertices. Furthermore, there exists a positive constant a_2 such that, for every $\epsilon \leq \frac{1}{2}$ and every s, there exists a planar graph G of radius no more than s with non-negative vertex weights summing to no more than 1 such that, for every set C of vertices whose deletion from G leaves only components of weight no more than ϵ, C must have at least $a_2 \cdot s/\epsilon$ vertices.*

Proof. The sufficiency follows from the construction of Theorem 1. To show necessity in the worst case, suppose for the sake of contradiction that for every constant a_2, there exists an $\epsilon \leq \frac{1}{2}$ (sufficiently small) and s (sufficiently large) such that, for every planar graph G of radius no more than s with non-negative vertex weights summing to no more than 1, there exists a set C of vertices whose deletion from G leaves only components of weight no more than ϵ such that $|C| < a_2 \cdot (s/\epsilon)$.

Let a_2 be chosen to be less than $(a_1/2)^2$ where a_1 is the constant mentioned in the proposition above. Consider a grid graph with n vertices where n is the smallest integer such that $n \cdot \epsilon > a_2 s^2$. Suppose that we have $\sqrt{n} \geq s$. Then, by our assumption and by using the construction of Theorem 1, we can show that there exists a set C with $n/s + (s/\epsilon) \cdot a_2$ vertices whose deletion separates the graph into components of weight no more than ϵ. However, by the proof of proposition above, $n/s + (s/\epsilon) \cdot a_2 \geq a_1\sqrt{n/\epsilon}$. By the choice of n, we then have that $2\sqrt{n/\epsilon}\sqrt{a_2} \geq a_1\sqrt{n/\epsilon}$. This implies that $a_2 \geq (a_1/2)^2$, a contradiction. On the other hand, suppose that we have $\sqrt{n} < s$. In this case, addition of suitable edges on one side of the grid yields a planar graph of radius $\sqrt{n}$ of which the grid is a subgraph. Then, by our assumption, there exists a set C with $(\sqrt{n}/\epsilon) \cdot a_2$ vertices whose deletion separates the graph into components of weight no more than ϵ. Again, by the proposition, we must have $(s/\epsilon) \cdot a_2 > (\sqrt{n}/\epsilon) \cdot a_2 \geq a_1\sqrt{n/\epsilon}$. This implies that $\sqrt{n/\epsilon}\sqrt{a_2} \geq a_1\sqrt{n/\epsilon}$, that is, $a_2 \geq (a_1)^2$, a contradiction. □

Observe that our goal here is to show a lower bound in the *worst case,* that is, *there exists* a planar graph of radius no more than s for which the deletion of at least $a_2 \cdot s/\epsilon$ vertices is necessary to leave only components of weight no more than ϵ.

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